Conditional entropy maximization for PET image reconstruction using adaptive mesh model

Hongqing Zhu a, Huazhong Shu a, b, *, Jian Zhou a, b, Xiubin Dai a, Limin Luo a, b

a Laboratory of Image Science and Technology, Department of Computer Science and Engineering, Southeast University, 210096 Nanjing, People’s Republic of China
b Centre de Recherche en Information Biomédicale Sino-français (CRIBs), France

Received 9 July 2005; received in revised form 11 September 2006; accepted 3 January 2007

Abstract

Iterative image reconstruction algorithms have been widely used in the field of positron emission tomography (PET). However, such algorithms are sensitive to noise artifacts so that the reconstruction begins to degrade when the number of iterations is high. In this paper, we propose a new algorithm to reconstruct an image from the PET emission projection data by using the conditional entropy maximization and the adaptive mesh model. In a traditional tomography reconstruction method, the reconstructed image is directly computed in the pixel domain. Unlike this kind of methods, the proposed approach is performed by estimating the nodal values from the observed projection data in a mesh domain. In our method, the initial Delaunay triangulation mesh is generated from a set of randomly selected pixel points, and it is then modified according to the pixel intensity value of the estimated image at each iteration step in which the conditional entropy maximization is used. The advantage of using the adaptive mesh model for image reconstruction is that it provides a natural spatially adaptive smoothness mechanism. In experiments using the synthetic and clinical data, it is found that the proposed algorithm is more robust to noise compared to the common pixel-based MLEM algorithm and mesh-based MLEM with a fixed mesh structure.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Image reconstruction; PET; Conditional entropy maximization; Adaptive mesh model

1. Introduction

Statistical-based iterative reconstruction algorithms have been widely used in the field of positron emission tomography (PET). This kind of algorithms take into account the stochastic properties of the measured data, and use accurate physical models for detector response, so that they can provide precise reconstructed images. The maximum likelihood-expectation maximization (MLEM) algorithm [1], which is a general statistical method for seeking an estimate of the image, allows computing the projections that are close to the measured projection data. Because the tomography reconstruction with a limited number of data is a highly underdetermined ill-posed problem, a solution of such problem is very sensitive to the level of noise. Since the projection data generated by the PET system are initially noisy, the MLEM algorithm tends to increase this noise and in particular the noise artifacts through the successive iterations. This accumulation of noise leads to a premature stopping of the MLEM reconstruction process [2]. To improve the quality of the reconstructed image, the regularization technique is often required. Recently, Bayesian reconstruction method has been used in PET imaging, it can effectively reduce the noise while preserving the image edge [3,4]. Another promising approach to the emission tomography problem is based on the conditional entropy maximization (CEM) algorithm. Both Bayesian algorithm and CEM are generalization of ML method [5,6], but the latter algorithm appears to be more suitable for tomography reconstruction [5,7–10]. This is probably due to the fact that the CEM estimation provides a consistent way of selecting a single image among the images that fit the measurement data [5]. Moreover, the Bayesian solution is equivalent to the maximum entropy for a special choice of the image prior probability density model [11].

In this paper, we investigate a new PET image reconstruction algorithm which combines the CEM and the adaptive mesh model. In the proposed method, we use the cross entropy as a prior in CEM algorithm because it is a powerful tool for solving the ill-posed inverse problem. The use of the entropy in
determining the prior probability has been discussed in several papers [7,9,12,13]. On the other hand, the mesh modeling of the images has been recently found wide application in image processing such as the image compression [14], image restoration [15], motion tracking [16] and medical image analysis [17,18]. Mesh modeling is an efficient and compact method for representing the image and tracking the rigid or non-rigid motion in image sequences. In a mesh model, the image domain is subdivided into a collection of mesh elements, whose vertices are called nodes. The whole image is then obtained by interpolation from the node values. The motivation for us to utilize the mesh model for emission reconstruction is that it can greatly alleviate the ill-posed nature of the reconstruction problem by reducing the number of unknowns, thus leads to improved quality in the reconstruction result. In addition, the reduction of the number of unknowns allows a fast computation [19,20]. Mesh model has been successfully applied to single photon emission computed tomography (SPECT) reconstruction by Brankov et al. [18]. In their algorithm, the image to be reconstructed was first modeled by an efficient mesh representation. The image was then obtained through the estimation of the node values from the measured data. Since the algorithm proposed in [18] is based on a fixed mesh model, the reconstruction result depends heavily on the initial mesh structure. It is difficult to select an appropriate initial mesh when we have not the prior information about the image to be reconstructed. The method proposed in [18] for determining the initial mesh is based on the filtered backprojection (FBP) image. It is known that the image reconstructed with FBP is characterized by a high level of noise inside and outside the object and the presence of streaky artifacts, the mesh model generated from such an image cannot perfectly describe the object feature. In order to solve these disadvantages, we propose an adaptive mesh-based algorithm for PET reconstruction. In our method, the Delaunay triangulation mesh is used in the adaptive mesh model. The Delaunay triangulation is one of the automated mesh generation algorithms that have recently gained popularity due to its ability to generate meshes of arbitrary geometry for both simply connected and multi-boundary domains. In recent years, many efforts have been devoted to the adaptive Delaunay triangulation mesh generation [21–23]. The adaptive mesh structure allows the proposed method more robust to noise compared to the reconstruction approach with fixed mesh model. Simulation studies with the Shepp–Logan phantom and clinical PET emission projection data show that the proposed algorithm outperforms the pixel-based MLEM and mesh-based MLEM methods.

The outline of this paper is as follows. In Section 2, we propose a pixel-based CEM PET reconstruction method. Section 3 describes the reconstruction method based on the adaptive mesh CEM. Section 4 provides some experimental results and compares the proposed method to other reconstruction algorithms. Section 5 concludes the work.

2. Pixel-based conditional entropy maximization reconstruction method

In PET, the isotope used emits positrons which annihilate with nearby electrons generating two photons traveling away from each other in opposite direction. The emission of the positrons is modeled as a spatial, inhomogeneous Poisson process with unknown intensity, the mean of which is determined by the concentration of the positrons that we want to estimate. We discretize the problem by subdividing the image to be reconstructed into $\sqrt{J} \times \sqrt{J}$ small square areas, also called pixels, and assume that the activity concentration within each pixel $j$ is uniform, denoted by $f_j$. The data consist of photon coincidence counts collected by $I$ detector pairs $y_i, i = 1, 2, \ldots, I$, $y_i$ is a sample from a Poisson distribution whose expected value is $E(Y_i) = (Af)_i = \sum_{j=1}^{J} a_{ij} f_j$, where $Y_i$ is the Poisson random variable corresponding to $y_i$ and $a_{ij}$ is the probability that a positron emitted at pixel $j$ will be detected by detector pair $i$. For the sake of simplicity, we assume that

$$\sum_{j=1}^{J} a_{ij} = 1 \quad (1)$$

Let $f = (f_1, \ldots, f_J)^T$ be the image vector and $y = (y_1, \ldots, y_I)^T$ the observed data vector where the superscript T denotes the transposition. The conditional probability for observing data $y$ given the emission parameter $f$ is the joint probability of the individual Poisson process expressed as

$$P\left(\frac{Y}{f}\right) = \prod_{i=1}^{I} \left[ \exp\left(\sum_{j=1}^{J} f_j a_{ij} \right) \frac{\left(\sum_{j=1}^{J} f_j a_{ij}\right)^{y_i}}{y_i!} \right] \quad (2)$$

The conditional entropy of the measurement process $y$ given the object image $f$ is defined by

$$H\left(\frac{Y}{f}\right) = -\sum_{f} P(f) \left[ \sum_{y} P\left(\frac{Y}{f}\right) \log P\left(\frac{Y}{f}\right) \right] \quad (3)$$

where $P(f)$ is a prior estimate of the image $f$.

The conditional entropy measures the information content of $y$ when image $f$ is known and the aim is to maximize the information content in the data set when the object image is given. To achieve this, we need to calculate the first derivative of Eq. (3) which is given by

$$\frac{\partial H(y|f)}{\partial f_j} = -\frac{\partial P(f)}{\partial f_j} \left[ \sum_{y} P\left(\frac{Y}{f}\right) \log P\left(\frac{Y}{f}\right) \right] - P(f) \frac{\partial}{\partial f_j} \left[ \sum_{y} P\left(\frac{Y}{f}\right) \log P\left(\frac{Y}{f}\right) \right] \quad (4)$$

The Kuhn–Tucker necessary and sufficient conditions for a maximum of $H(y|f)$ are then

$$\frac{\partial H(y|f)}{\partial f_j} = 0 \quad \text{for} \quad f_j > 0 \quad (5)$$

$$\frac{\partial H(y|f)}{\partial f_j} \leq 0 \quad \text{for} \quad f_j = 0 \quad (6)$$

The solution of the pixel-based CEM method for computing the estimated image $f$ can be obtained by the following iterative
relation (the detailed derivation is given in Appendix A):
\[
f^{(k+1)}_j = \frac{f^{(k)}_j}{P(f^{(k)}_j)\sum_{l=1}^N a_{lj}} \left[ \frac{\partial P(f_j)}{\partial f_j} \right]_{f_j = f^{(k)}_j} \\
+ P(f^{(k)}_j) \frac{\sum_{l=1}^N y_{lj}^2}{\sum_{l=1}^N f^{(k)}_l a_{lj}} , \quad j = 1, \ldots, J
\]

where \(k\) denotes the iteration number.

In [5], Mondal and Rajan found that the CEM algorithm approaches the MLEM algorithm if the prior function \(P(f)\) tends to a uniform distribution. Generally, a common Bayesian prior is a Gibbs distribution of the form:
\[
P(f) = \frac{1}{Z} \exp[-\beta U(f)]
\]
where \(Z\) is the normalizing constant for the distribution, \(\beta\) is a positive hyper-parameter, and \(U(f)\) is the energy function. In this paper, we select the cross entropy as energy function, which is defined as
\[
U(f) = \sum_i f_i \log \left( \frac{f_i}{\bar{f}_j} \right)
\]
where \(\bar{f}\) is the prior image model for PET image.

The specification for the prior model \(\bar{f}\) is of great importance. Ardekani [24] suggested two typical choices of \(\bar{f}\): The first one can be achieved dynamically, i.e., for each iteration \(k\), \(\bar{f}\) takes the filter version of \(\bar{f} = \tilde{f}\). The other is to incorporate the anatomical information obtained from magnetic resonance (MR) images of the same patient. In this paper, the first choice is adopted, that is, the image estimated from the previous iteration is used as the prior image model.

Note that the pixel-based CEM reconstruction algorithm, i.e., the method based on Eq. (7), updates all the pixels of the image at each iteration step. Moreover, the number of pixels used to estimate the image remains invariant during the whole iterative process.

### 3. Mesh-based conditional entropy maximization reconstruction algorithm

#### 3.1. Fixed mesh-based CEM algorithm

In [15], Brankov et al. represented an image function as
\[
f(x) = \sum_{n=1}^N f(x_n)\phi_n(x) + e(x)
\]
where \(f(x) = f(u,v)\) denotes an image function defined over a 2D domain \(D\), which is divided into a total of \(M\) non-overlapping mesh elements \(D_m\), \(m = 1, 2, \ldots, M\). \(x_n\) is the \(n\)th mesh node, \(e(x)\) is the interpolation error. \(\phi_n(x)\) is the interpolation basis function associated with the \(n\)th node \(x_n\) of \(D_m\), and \(N\) is the total number of mesh nodes.

Let \(q\) be the vector formed by the node values of the mesh model, and \(\phi(x)\) be the vector formed by the interpolation basis function, i.e.,
\[
q = [f(x_1), f(x_2), \ldots, f(x_N)]^T \quad (11)
\]
\[
\phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)]^T \quad (12)
\]
Then Eq. (10) can be rewritten as
\[
f(x) = \phi^T(x)q + e(x)
\]
Let \(f(x)\) denote the pixel representation of the image function \(f(x)\) over \(D\), then from Eqs. (11) and (13), one can obtain
\[
f(x) = \phi^T(x)q + e(x)
\]
where \(\phi\) is the interpolation matrix in which each row consists of the vector \(\phi^T(x)\), and \(e(x)\) is the vector representing the error \(e(x)\).

For the pixel-based image reconstruction algorithm, the imaging equation is written in terms of the pixel representation \(f\) as
\[
E(y) = Af
\]
where \(A\) is the projection matrix describing the imaging system, and \(y\) is the measured projection data.

Substitution of Eq. (14) into Eq. (15) yields the following mesh-domain imaging equation
\[
E(y) = (A\phi)q + Ae
\]
If we neglect the influence of noise, then Eq. (16) becomes
\[
E(y) \approx Bq
\]
where \(B = A\phi\), and the elements of the matrix \(B = \{b_{in}\} \in R^T \times N\) are given by
\[
b_{in} = \int_D a_i(x)\phi_n(x) \, dx
\]
where \(a_i\) is the \(i\)th row of the probability matrix, and \(a_i(x)\) denotes the probability of a positron emitted at location \(x\) detected by detector pair \(i\).

In the mesh-based reconstruction algorithm, the reconstruction problem consists of estimating \(q\) from the given \(y\). The mesh-domain model equation (17) has the similar form as that of the pixel-domain model equation (15). The difference between the two reconstruction algorithms relies on the probability matrix. In the mesh domain, the system probability matrix is calculated by Eq. (18) which is the integration over those individual elements attached to the node \(n\). Thus, if we neglect the interpolation error \(e(x)\), according to Eq. (7), we obtain the mesh-based conditional entropy maximization reconstruction method as follows:
\[
f^{(k+1)}(x_n) = \frac{f^{(k)}(x_n)}{P(f^{(k)}(x_n))\sum_{l=1}^M a_{ln}} \left[ \frac{\partial P(f(x_n))}{\partial f(x_n)} \right]_{f(x)=f^{(k)}(x_n)} \\
+ P(f^{(k)}(x_n)) \frac{1}{\sum_{l=1}^M f^{(k)}(x_l) b_{ln}} \sum_{l=1}^M y_{ln} b_{ln}
\]
Eq. (19) indicates that the mesh-based image reconstruction approach is similar in form to the common pixel-domain model.
Besides the system probability matrix, another difference is that for the former, the image is obtained through estimation of the node values from the measured projection data, and for the pixel-domain model, the reconstructed image is obtained by estimating all the pixel values from the projection data.

3.2. Adaptive mesh-based CEM algorithm

The use of the fixed mesh-based model could lead to good reconstruction result if the initial mesh structure is appropriately chosen. However, this is not a trivial task. In [18], Brankov et al. used a FBP image to generate a fixed mesh. Because a typical FBP image is characterized by a high level of noise inside and outside the object and the presence of the streaky artifacts, such a fixed mesh cannot perfectly describe the object feature. This motivates us to introduce an adaptive mesh-based algorithm for PET reconstruct. Since our method updates the mesh structure at each iteration, the choice of the initial mesh is not crucial. It is because according to [5], the entropy maximization algorithm for PET Eq. (7), which recursively produces a series of reconstructed images that converge to the maximum entropy estimate. Thus, although one randomly selects some pixel points as initial starting points to generate a rough Delaunay triangulation mesh. The intensity of each nodes of Delaunay triangulation mesh is estimated by using Eq. (19). The reconstructed image is then obtained by calculating each pixel value from the node values by linear interpolation. The whole reconstruction process of the proposed algorithm is described as follows:

- **Step 1:** Initialization of the mesh structure
  Instead of using the FBP image to generate the initial mesh as proposed in [15], which may be time extensive, we randomly select a set of pixel points, and a constant value is set to these pixels. This set of points is used to generate the initial Delaunay triangulation mesh.
- **Step 2:** Update of the image space
  The generated mesh structure serves as the basis for a customized basis representation of the image. The intensity value of each node is estimated from the measured projection data with the CEM algorithm described by Eq. (19). The whole image is then obtained by calculating each pixel value from the node values by linear interpolation.
- **Step 3:** Update of the mesh structure
  Once the image is reconstructed, the new mesh for the next iteration is modified according to the method described in [15].

(1) Feature map extraction

The feature map function $\sigma(u, v)$ is first extracted from the estimated image $f(u, v)$:

$$\sigma(u, v) = \left( \frac{G(u, v)}{B} \right)^r$$  \hspace{1cm} (20)

where $r$ is a parameter used to adjust the mesh structure. The smaller of the value of $r$, more number of nodes located at the boundary region of the image will be. $B$ is the largest value of $G(u, v)$ over the image domain, it is used to normalize the value $\sigma(u, v)$ within the range $[0, 1]$. $G(u, v)$ is given by

$$G(u, v) = \max(\{\lambda_1(u, v), \lambda_2(u, v)\})$$  \hspace{1cm} (21)

where $\lambda_{1,2}(u, v)$ represent the two eigenvalues of the Hessian matrix of $f(u, v)$ at $(u, v)$, and can be computed as follows:

$$\lambda_{1,2}(u, v) = \frac{1}{2} \left( f''_{uu}(u, v) + f''_{uv}(u, v) \right)$$

$$\pm \sqrt{\frac{1}{4} \left( f''_{uu}(u, v) - f''_{uv}(u, v) \right)^2 + \left( f''_{uv}(u, v) \right)^2}$$  \hspace{1cm} (22)

where $f''_{uu}(u, v)$, $f''_{uv}(u, v)$ and $f''_{uv}(u, v)$, and are the second-order of the partial derivatives of $f$.

(2) Location of the mesh nodes

The mesh node location is determined from the feature map using the Floyd–Steinberg error-diffusion algorithm [25]. We utilize the Floyd–Steinberg algorithm to place the mesh nodes in accordance with the density specified by the feature map. The reader are referred to [25] for more detail of this algorithm. Fig. 1 shows the pseudo code to describe the error diffusion. Note that in Fig. 1, $B(u, v) = 1$ indicates the presence of a mesh node at $(u, v)$. The mesh nodes are placed at those pixels with nonzero entries in the matrix $[B(u, v)]_{u=\text{max}, v=\text{max}}$. The total number of the mesh nodes, denoted by $K$, can be

/* The order of pixel visitation generally takes the form of raster processing: */
for (u = 0; u < u_max; u++) {
  for (v = 0; v < v_max; v++) {
/* for every pixel position u, v in feature map function */
    if ($\sigma(u, v) < q$
      then $B(u, v) = 0$
      else $B(u, v) = 1$
    error = $\sigma(u, v) - 2q \cdot B(u, v)$
/* distribute the error among unprocessed neighbors of u, v */
    $\sigma(u, v+1) += \text{error} \times 7/16$
    $\sigma(u+1, v) -= \text{error} \times 3/16$
    $\sigma(u+1, v+1) += \text{error} \times 5/16$
    $\sigma(u+1, v-1) += \text{error} \times 1/16$
  }
}

Fig. 1. Pseudo-code for describing the error diffusion.
computed as follows:

\[ K = \sum_{u=1}^{u_{\text{max}}} \sum_{v=1}^{v_{\text{max}}} B(u, v) \]  

(23)

Using the relation error \( e = \sigma(u, v) - 2qB(u, v) \) given in Fig. 1, Eq. (23) becomes

\[ K = \frac{1}{2q} \sum_{u=1}^{u_{\text{max}}} \sum_{v=1}^{v_{\text{max}}} [\sigma(u, v) - e(u, v)] \]  

(24)

(3) Generation of the new mesh

The mesh structure can be obtained by connecting the mesh nodes by means of Delaunay triangulation. An interesting property of the Delaunay triangulation is that it connects a given set of mesh nodes in such a way that the circle circumscribing any triangular element contains only the nodal points belonging to that triangle (except for the case where four or more nodal points are co-circular). The Delaunay triangulation allows generating a well-structured mesh at a reasonable computational cost, and avoids producing excessively elongated elements [26,27].

- **Step 4**: Stop if the maximum number of iterations is reached. Return to Step 2 if not.

The potential benefit of using the adaptive mesh image reconstruction approach is that it provides a natural spatially adaptive smoothness mechanism. A key feature in this mesh model is that it uses non-uniform sampling, of which the mesh nodes are adaptively placed according to the local content of the image at each iteration, and the interpolation basis functions used in the mesh model have the spatial support varying with the local image content. More precisely, smaller mesh elements are placed at the regions containing high frequency features, while larger elements are placed in the regions containing low frequency components [15]. Since the estimated image of each iteration is computed on every triangular element using the linear interpolation, the influence of the noise on the reconstructed image can be effectively reduced because of the smoothness generated by the linear interpolation method. This is particularly suitable for PET reconstruction because the projection data are usually noisy.

4. Experimental results and discussion

In the simulation experiments, we first use a 96 × 96 pixels Shepp–Logan phantom obtained from the website [28] to test the feasibility of the proposed method. The relative activities in different regions of the phantom are shown in Fig. 2. The simulated projections were calculated from such geometrical definitions as was the discrete representation of the phantom. The projection space is assumed to be 96 bins and 96 angular views evenly spaced over 180°. The final reconstructed image is set to a size of 96 × 96 pixel matrix. Poisson noise was added to the simulated projections to make the situation more realistic. The noisy projection data were generated by adding to the original projections a uniform field of random coincidences reflecting a scan of 25% of the total count. Complicating factors such as attenuation and scatter were not considered. In this example, projections

with Poisson noise are used in all the reconstruction methods. The normalizing constant \( Z \) is set to 1 and will be kept in the CEM approach whether the reconstruction is performed in the pixel domain or in the mesh domain. The accuracy and the convergence of the reconstructed results towards the true phantom values were evaluated by means of the variance. It is computed using the formula below

\[ \sigma = \sqrt{\frac{1}{J-1} \sum_{j=1}^{J} (f_{j}^{\text{rec}} - f_{j}^{\text{org}})^2} \]  

(25)

where \( f_{j}^{\text{org}} \) is the average gray level of original Shepp–Logan phantom, and \( f_{j}^{\text{rec}} \) denotes the value of the reconstructed image at pixel \( j \).

Fig. 3 depicts the reconstruction images using the pixel-based MLEM and CEM reconstruction approaches after running 40 iterations. It can be observed that the noise contaminates the MLEM image at high iteration numbers. The error analysis for the reconstruction images is presented in Fig. 4. Comparison of the variances of the two methods shows that the reconstruction image with CEM algorithm outperforms the MLEM image in terms of the contrast and the discriminability. This is because the use of the prior term (also called regularization term) in the CEM algorithm allows reducing the noise artifacts [5,7]. Due to the tomography reconstruction with a limited number of data appears as a highly underdetermined ill-posed problem, images reconstructed purely by using the ML criterion are characterized by noise artifacts [29]. Methods for reducing the noise in reconstructed images mainly include: premature stopping rules [2], various regularized technologies, such as penalized least
squares [30], smoothing steps [31], Green’s one step late algorithm [3]. Similarly, pixel-based CEM reconstruction also uses the prior distribution knowledge of the emission densities in the reconstruction process which make it outperform over the purely MLEM algorithm [5].

In the reconstruction with CEM algorithm, the hyperparameter $\beta$ appeared in Eq. (8) may affect the reconstruction results. The $\beta$ value indicates the amount of the prior to be introduced into the reconstruction. If $\beta$ reaches a certain value, then the reconstruction is effective due to the likelihood contribution [6]. We find that $\beta \in [1 \times 10^{4}, 1 \times 10^{5}]$ provides the best reconstruction, so that $\beta = 2.5 \times 10^{4}$ is chosen for the CEM algorithm in the remaining of the paper.
Fig. 7. The reconstructed errors of the adaptive mesh-based algorithms: (a) MLEM; (b) CEM.

Fig. 8. (a) Initial mesh structure, 478 nodes used; (b) the final mesh structure obtained by using the adaptive mesh MLEM, $r=0.8$, 401 nodes used; (c) the final mesh structure obtained by using the adaptive mesh MLEM, $r=0.5$, 642 nodes used; (d) reconstructed image using the adaptive mesh MLEM, $r=0.8$; (e) reconstructed image using the adaptive mesh MLEM, $r=0.5$; (f) the final mesh structure obtained by using the adaptive mesh CEM, $r=0.8$, 406 nodes used; (g) the final mesh structure obtained by using the adaptive mesh CEM, $r=0.5$, 649 nodes used; (h) reconstructed image using the adaptive mesh CEM, $r=0.8$; (i) reconstructed image using adaptive mesh CEM, $r=0.5$. 
this figure that the reconstruction error is important for both methods when the number of nodes is high (e.g. 6197 nodes for $q=0.2$ and $r=0.3$). Based on the test results, we found that the quantization parameter $q=0.5$, $r=0.8$ (or $q=0.5$, $r=0.5$) is an adequate choice, so that this value will be used for the mesh-based reconstruction algorithms. Fig. 5(b) and (c) shows the two mesh structures with the parameter $r$ being 0.8 and 0.5, respectively. These two meshes have 1051 and 1813 mesh nodes (about 11.4% and 19.7% of the total number of pixels in the phantom), respectively. The corresponding reconstruction results using mesh-based MLEM algorithm are shown in Fig. 5(d) and (e), respectively. Note that the maximum number of iterations is fixed to 40 except for FBP method. For comparison purpose, we also show the reconstructed images obtained with the fixed mesh-based CEM algorithm. Fig. 5(f) and (g) shows the reconstructed results. As can be seen from these figures, the noise accumulation in PET image can be reduced by both mesh-based MLEM and CEM methods. From a visual standpoint, one can observe that mesh-based CEM reconstructed images are smooth. Contrastively, we see that the noise was removed from the interior regions of mesh-based MLEM but it is still present along the edges, which result in slight artifacts around the edges. Because of the ill posedness of the reconstruction problem, a penalty is often imposed on the solution to control noise. The difficulty is to preserve the edges, which are very important attributes of the image. From Fig. 5, we found that the mesh-based CEM reconstruction with cross entropy prior can comparatively reduce the edge artifacts.

We then compare the reconstructed images of the adaptive mesh-based MLEM and adaptive mesh-based CEM algorithms. The initial Delaunay triangulation mesh structure generated by 478 randomly selected pixel points is presented in Fig. 8(a). This mesh structure is then automatically adjusted according to the currently estimated image at each iteration step. The dense mesh elements are placed at the regions containing high-frequency features of the image. Fig. 8(b) and (c) shows the final mesh structure after 40 iterations with different values of the parameter $r$ (Note that $q=0.5$ is used). The mesh nodes are reduced from 1051 to 401 for $r=0.8$ and from 1813 to 642 for $r=0.5$, respectively. The corresponding MLEM reconstruction results of the phantom with the adaptive mesh model for different values of $r$ are shown in Fig. 8(d) and (e) after 40 iterations. The phantom sinogram is also reconstructed with the adaptive mesh-based CEM algorithm described by Eq. (19), Fig. 8(f) and (g) shows the final mesh structure with adaptive mesh CEM algorithm. The corresponding reconstruction images for $r=0.8$ and $r=0.5$ are depicted in Fig. 8(h) and (i), respectively. The results show that the adaptive mesh-based CEM algorithm is more suitable to reduce the noise artifact. We have also found that for the fixed values of $q$ and $r$, the final mesh structure obtained with the adaptive mesh-based CEM algorithm is very similar to that of the adaptive mesh-based MLEM algorithm whatever the initial mesh is used. Table 1 lists the number of the final mesh nodes of both methods for the example shown in Fig. 8. This interesting phenomenon indicates that the adaptive mesh-based algorithms should be independent of the initial mesh.

A quantitative analysis of the reconstruction error is also performed for the fixed mesh MLEM, fixed mesh CEM, adaptive mesh MLEM, and adaptive mesh CEM algorithm. Fig. 9(a) shows that the standard deviation values using the adaptive mesh CEM with parameter $r=0.8$ and $q=0.5$ decrease faster than those obtained with other methods. The same phenomenon can be observed in Fig. 9(b) in which $r=q=0.5$ is used for the adaptive mesh CEM algorithm. The results show that the use of the adaptive mesh technique allows reducing the noise artifact.

Finally, we apply the algorithms to reconstruct the clinical PET emission projections obtained from a Siemens Biograph Sensation 16 CT/PET scanner. Emission data were obtained after

---

### Table 1

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>Initial nodes is 478</th>
<th>Initial nodes is 313</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r=0.8$</td>
<td>$r=0.5$</td>
</tr>
<tr>
<td>Adaptive mesh MLEM</td>
<td>401</td>
<td>405</td>
</tr>
<tr>
<td>Adaptive mesh CEM</td>
<td>406</td>
<td>649</td>
</tr>
</tbody>
</table>

---

![Fig. 9. The comparison results of the mesh MLEM, mesh CEM, adaptive mesh MLEM, and adaptive mesh CEM methods using the noisy projections: (a) $r=0.8$; (b) $r=0.5$.](image-url)
the injection of tracer $^{18}$F FDG. For the 16-slice spiral CT scan, the following settings were used: 0.125 pitch, 3.0 mm/s table feed, 0.5 s rotation time, 16 mm $\times$ 1.5 mm scanning slice thickness, 3 mm reconstructed slice thickness and 500 mm field of view (FOV). The PET data were obtained in list mode using a list mode research package originally provided by Siemens. List mode data were acquired for 10 min. We obtained this original emission data in DICOM format from Siemens' software. Then, we transformed them to raw data. The reconstructed PET image size was $128 \times 128$. The emission data resulted in a sinogram of size $192 \times 192$.

Reconstructions of the real clinical data with the pixel MLEM and pixel CEM after 40 iterations are shown in Fig. 10(a) and (b), respectively. Fig. 10 shows that the image obtained with the pixel-based MLEM is much more noisy than that of the pixel CEM method.

Fig. 11(a) shows the result with the conventional FBP image. It is clear that the FBP reconstruction is characterized by noise and cannot effectively recover the sharp edges. Fig. 11(b) depicts the mesh structure generated directly from this FBP image. The number of the mesh nodes used is 1040 (about 6.34% of the number of pixels in the reconstruction image). The images shown in Fig. 11(c) and (d) are obtained with the fixed mesh-based MLEM and CEM algorithms, respectively. From Fig. 11, it can be seen that the fixed mesh-based MLEM and CEM algorithms reduce well the noise in the reconstructed images. The form of two high bright spots of larynx can be clearly seen in Fig. 11(d) where the two red arrows indicate the high metabolism of the membrana elastica larynx.

The adaptive mesh-based MLEM and CEM approaches are also applied to this example. Fig. 12(a) shows the initial mesh structure containing 990 nodes. The final mesh structure and reconstructed result with the adaptive mesh MLEM algorithm for $r=0.8$ are shown in Fig. 12(b) and (c), respectively. The number of the mesh nodes reduces to 814. Fig. 12(d) depicts the final mesh model of the adaptive mesh-based CEM algorithm where the number of nodes is 822, and Fig. 12(e) shows the corresponding reconstruction result.
5. Summary

We have presented a method for PET reconstruction based on the conditional entropy maximization and adaptive mesh model. Because the artifacts appeared in FBP reconstructed image are difficult to be removed, and the fixed mesh generated from such a FBP image cannot perfectly describe the object feature, we have used a set of the pixel points randomly selected to generate the initial Delaunay triangulation mesh structure. The proposed approach adaptively adjusts the Delaunay triangulation mesh according to the estimated image intensity during the iteration process in which the conditional entropy maximization is used. The experimental results show that the method is effective, and robust to noise, which indicate that the adaptive mesh model could be a useful technique to improve the image quality for PET reconstruction.

Acknowledgements

This work was supported by National Basic Research Program of China under grant No. 2003CB716102 and Program for New Century Excellent Talents in University under grant No. NCET-04-0477. It has been carried out in the frame of the CRIBs, a joint international laboratory associating Southeast University, the University of Rennes 1 and INSERM, with a grant provided by the French Consulate in Shanghai. We thank the anonymous referees for their careful review and valuable comments to improve the quality of the paper.

Appendix A

According to Eq. (4), the first derivative of the conditional entropy is given by

\[
\frac{\partial H(y/f)}{\partial f_j} = -\frac{\partial P(f)}{\partial f_j} \left[ \sum_y P(y/f) \log P(y/f) \right] - \left[ P(f) \frac{\partial}{\partial f_j} \left( \sum_y P(y/f) \log P(y/f) \right) \right]
\]

(Eq. A1)

Eq. (A1) can be rewritten as

\[
\frac{\partial H(y/f)}{\partial f_j} = \sum_y \left[ P(y/f) \log P(y/f) \frac{\partial P(f)}{\partial f_j} \right] + P(f) \left( \log P(y/f) \frac{\partial P(y/f)}{\partial f_j} \right)
\]

\[+ P(f) \left( 1 + \log P(y/f) \right) \frac{\partial}{\partial f_j} \left[ \log P(y/f) \right] \]

(A2)

The Kuhn–Tucker necessary and sufficient conditions for a maximum of \(H(y/f)\) are as follows:

\[
\frac{\partial H(y/f)}{\partial f_j} = 0 \quad \text{for} \quad f_j > 0
\]

(A3)

\[
\frac{\partial H(y/f)}{\partial f_j} \leq 0 \quad \text{for} \quad f_j = 0
\]

(A4)

thus

\[
\frac{\partial P(f)}{\partial f_j} \left( P(y/f) \log P(y/f) \right) + P(f) \left( 1 + \log P(y/f) \right) \frac{\partial}{\partial f_j} \left[ \log P(y/f) \right] = 0
\]

(A5)

According to Eq. (2), we have

\[
\frac{\partial}{\partial f_j} \left[ \log P(y/f) \right] = \frac{\partial}{\partial f_j} \log \prod_{i=1}^{I} \left[ \exp \left( -\sum_{j=1}^{J} f_j a_{ij} \right) \left( \sum_{j=1}^{J} f_j a_{ij} \right)^{y_{ij}} \right]
\]

\[= \frac{\partial}{\partial f_j} \left[ \prod_{i=1}^{I} \left( -\sum_{j=1}^{J} f_j a_{ij} \right) \left( \sum_{j=1}^{J} f_j a_{ij} \right)^{y_{ij}} \right]
\]

\[= \frac{\partial}{\partial f_j} \left[ \prod_{i=1}^{I} \left( \sum_{j=1}^{J} f_j a_{ij} + y_{ij} \log \sum_{j=1}^{J} f_j a_{ij} - \log(y_{ij}) \right) \right]
\]

(A7)

Substitution of Eq. (A7) into Eq. (A5), we obtain

\[
P(y/f) \left[ \frac{\partial P(f)}{\partial f_j} \log P(y/f) + P(f) \left( 1 + \log P(y/f) \right) \right]
\]

\[\times \sum_{i=1}^{I} \frac{y_{ij} a_{ij}}{\sum_{j=1}^{J} f_j a_{ij}} - P(f) \left( 1 + \log P(y/f) \right) \sum_{i=1}^{I} a_{ij} \right] = 0
\]

(A8)
Since $P(y|x) \neq 0$, we obtain the fixed point iteration equation as follows:

$$f_{j}^{(k+1)} = \frac{f_{j}^{(k)}}{P(f_{j}^{(k)})} \left( 1 + \log P(y|f_{j}^{(k)}) \right) \sum_{i=1}^{l} \alpha_{ij}$$

$$\times \left[ \frac{\partial P(f_{j})}{\partial f_{j}} \bigg|_{f_{j}=f_{j}^{(k)}} \log P \left( \frac{y}{f_{j}^{(k)}} \right) + P(f_{j}^{(k)}) \left( 1 + \log P \left( \frac{y}{f_{j}^{(k)}} \right) \right) \right]$$

$$+ \frac{P(f_{j}^{(k)})}{I} \left( 1 + \log P \left( \frac{y}{f_{j}^{(k)}} \right) \right) \sum_{i=1}^{l} \frac{y_{i} \alpha_{ij}}{\sum_{j=1}^{J} f_{j}^{(k)} \alpha_{ij}}$$

(A9)

Using the approximation

$$1 + \log P \left( \frac{y}{f_{j}^{(k)}} \right) \approx \log P \left( \frac{y}{f_{j}^{(k)}} \right)$$

(A10)

we deduce the following iterative relation:

$$f_{j}^{(k+1)} = \frac{f_{j}^{(k)}}{P(f_{j}^{(k)})} \sum_{i=1}^{l} \alpha_{ij}$$

$$\times \left[ \frac{\partial P(f_{j})}{\partial f_{j}} \bigg|_{f_{j}=f_{j}^{(k)}} \log \left( \frac{y_{i} \alpha_{ij}}{\sum_{j=1}^{J} f_{j}^{(k)} \alpha_{ij}} \right) \right], \ j = 1, \ldots, J$$

(A11)

References


Hongqing Zhu obtained her Ph.D. degree in 2000 from Shanghai Jiao Tong University. From 2003 to 2005, she was a postdoctoral fellow in the Department of Biology and Medical Engineering of Southeast University. Now, she is an associate professor in East China University of Science & Technology. Her current research is mainly focused on image reconstruction, image segmentation, image compression, and pattern recognition.

Huaizong Shou received the B.S. degree in applied mathematics from Wuhan University, China, in 1987, and a Ph.D. degree in numerical analysis from the University of Rennes (France) in 1992. He was a postdoctoral fellow in the Department of Biology and Medical Engineering, Southeast University, from 1995 to 1997. He is now with the Department of Computer Science and Engi-
neering of the same university. His recent work concentrates on the treatment planning optimization, medical imaging, and pattern recognition.

Jian Zhou received the B.S. and M.S. degree in radio engineering from Southeast University, China, in 2000 and 2003, respectively. He is now pursuing his Ph.D. degree in the Department of Biology and Medical Engineering of Southeast University. His current research is mainly focused on pattern recognition and image processing.

Xiubin Dia received the B.S. degree in the Department of Biology and Medical Engineering in 2003 from Southeast University and now is a graduate student of this Department. Her current research is mainly focused on pattern recognition and image processing.

Limin Luo obtained his Ph.D. degree in 1986 from the University of Rennes (France). Now he is a professor of the Department of Computer Science and Engineering, Southeast University, Nanjing, China. He is the author and co-author of more than 80 papers. His current research interests include medical imaging, image analysis, computer-assisted systems for diagnosis and therapy in medicine, and computer vision. He is a senior member of the IEEE. He is an associate editor of IEEE Eng. Med. Biol. Magazine and Innovation et Technologie en Biologie et Medecine (ITBM).